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Brane-inspired orientifold field theories

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Abstract: In this paper we consider the gauge theory living on the world-volume of a stack of N D3-branes of type $0\mathrm{B}/\Omega' I_6(-1)^{F_L}$ and of its orbifolds $C^2/\mathbb{Z}_2$ and $C^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$. The gauge theories obtained in the three cases are a brane realization of "orientifold field theories" having the bosonic sector common with $\mathcal{N}=4,2,1$ super Yang-Mills respectively. In these non-supersymmetric theories, we investigate the possibility of keeping the gauge/gravity correspondence that has revealed itself so successful in the case of supersymmetric theories. In the open string framework we compute the coefficient of the gauge kinetic term showing that the perturbative behaviour of the orientifold field theory can be obtained from the closed string channel in the large N limit, where the theory exhibits Bose-Fermi degeneracy.

Keywords: D-branes, String Duality.

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1. Introduction

In the last years it has been shown in several cases that the pertubative properties such as the chiral and scale anomalies [1]–[3] of less supersymmetric and non-conformal theories living on D-branes can be obtained from their correponding supergravity solutions.[1] This came as a surprise because the supergravity "dual" solution was supposed to give a correct description of the gauge theory for large values of the 't Hooft gauge coupling constant [10]. The explanation of this fact was given in ref. [11] where it was shown, in the case of the two orbifolds $C^2/\mathbb{Z}_2$ and $C^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, that the contribution of the massless open string states to the coefficient of the gauge kinetic term obtained from the annulus diagram is exactly equal, under open/closed string duality, to the contribution of the massless closed string states. Actually it can also be shown that the contribution of the massive states is identically zero giving no threshold corrections [12].[2]

The previous results have been shown to be valid in the case of theories that, although non-conformal, preserve some supersymmetry. Can they be extended to non-supersymmetric theories? Obvious candidates for non-supersymmetric theories are the type 0 ones that have been studied by constructing their supergravity duals in refs. [13]–[15]. The problem is, however, that they have a tachyon in the closed string sector. Tachyon free

[1] For general reviews on various approaches see refs. [4]–[8]. See also ref. [9]

[2] We thank Jose' F. Morales for pointing this out to us.

orientifolds of these theories, called $0'$ theories, were introduced in Ref. [16] and their properties were extensively studied from different points of view in refs. [17]–[19]. On the other hand it is clear that the most promising theories to study with these methods are the ones that are as close as possible to supersymmetric theories. Recently non-supersymmetric theories have been studied that in the large number of colours are equivalent to supersymmetric theories.[3] In refs. [16, 18, 22], non-supersymmetric gauge theories that are conformal in the planar limit have been analyzed. In particular a large N conformal non-supersymmetric gauge theory may be obtained as the world-volume theory of N D3-branes of the orientifold $\Omega' I_6(-1)^{F_L}$ of 0B theory, where Ω' is the world-sheet parity, I_6 the inversion of the coordinates orthogonal to the world-volume of the D3-branes and F_L is the space-time fermion number operator in the left sector. The gauge theory so constructed is an example of "orientifold field theory" which in the large N limit is equivalent to $\mathcal{N}=4$ super Yang-Mills.

More recently some attention has been paid to the orientifold field theories that contain a gluon and a fermion transforming according to the two-index symmetric or antisymmetric representation of the gauge group SU(N) [23] and that in the large N limit are equivalent to $\mathcal{N}=1$ SYM.

In this paper we review the brane construction of the $\mathcal{N}=4$ orientifold field theory, which is planar equivalent (i.e. equivalent in the limit $N\to\infty$ with $\lambda = g^2_{YM}N$ fixed) to $\mathcal{N}=4$ super Yang-Mills. By means of the orbifold projections $C^2/\mathbb{Z}_2$ and $C^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$, we give a complete string description of the orientifold field theory whose spectrum has, in the large N limit, the same number of degrees of freedom as $\mathcal{N}=2,1$ super Yang-Mills. The latter theory has been shown to be planar equivalent, both at perturbative and not-perturbative level, to $\mathcal{N}=1$ SYM [20].

In the open string framework we compute the running coupling constant and show that in the large N limit, where the Bose-Fermi degeneracy of the gauge theory is recovered, we can obtain the perturbative behaviour of the orientifold field theories also from the closed string channel. We see, however, that the next to leading term in the large N expansion of the β-function cannot be obtained from the closed string channel. This means that gauge/gravity correspondence holds only in the planar limit as far as the running coupling constant is concerned. When we then consider the θ-angle we see that both the leading and the next to leading terms can be equivalently determined from the open and the closed string channel. This follows from the fact that in the string framework the θ-angle does not admit threshold corrections and thus it is invariant under open/closed string duality.

The paper is organized as follows. In section 2 we summarize the main properties of type 0 theories. Section 3 is devoted to the construction of the orientifold $0B/(\Omega' I_6(-1)^{F_L})$. In the first subsection we construct its open and closed string spectrum, in the second one we compute the one-loop open string diagrams and finally in the third one we introduce an external field [24] and we compute the running coupling constant in both the open and closed string channels finding agreement between them only for large values of N, being

[3] See the recent review by Armoni, Shifman and Veneziano [20] and references therein. In ref. [21] $1/N$ corrections are analysed.

U(N) the gauge group. In section 4 we consider the orbifolds $C^2/\mathbb{Z}_2$ and $C^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ of the previous orientifold obtaining non-supersymmetric gauge theories that in the large N limit reduce respectively to $\mathcal{N}=2$ and $\mathcal{N}=1$ SYM. Finally in the appendix we perform some calculations useful in section 3.

2. Type 0B string theory

In this section we summarize the properties of type 0 string theories. In particular we will discuss their spectrum and D branes and compute the annulus diagram describing the interaction between D branes.

Type 0 string theories are non supersymmetric string models obtained by applying on the Neveu-Scharz-Ramond model for closed strings the following non-chiral diagonal projections:

$$P_{\mathrm{NS-NS}} = \frac{1+(-1)^{F+\tilde{F}+G+\tilde{G}}}{2} \qquad P_{\mathrm{R-R}} = \frac{1\pm(-1)^{F+\tilde{F}+G+\tilde{G}}}{2} \tag{2.1}$$

where the upper (lower) sign corresponds to 0B (0A). F is the world-sheet fermion number defined as

$$F = \sum_{t=1/2}^{\infty} \psi_{-t}\cdot\psi_t - 1 \tag{2.2}$$

in the NS-NS sector and

$$(-1)^F = \psi_{11}(-1)^{F_R}\,, \qquad \psi_{11} \equiv 2^5\psi_0^0\psi_0^1\ldots\psi_0^9\,, \qquad F_R = \sum_{n=1}^{\infty}\psi_{-n}\cdot\psi_n \tag{2.3}$$

in the R-R sector, with analogous definitions for $\tilde{F}$ and $(-1)^{\tilde{F}}$. G is the superghost number operator defined as follows:

$$G = -\sum_{m=1/2}^{\infty}(\gamma_{-m}\beta_m + \beta_{-m}\gamma_m) \qquad \text{NS sector} \tag{2.4}$$

$$G = -\gamma_0\beta_0 - \sum_{m=1}^{\infty}(\gamma_{-m}\beta_m + \beta_{-m}\gamma_m) \qquad \text{R sector}\,. \tag{2.5}$$

In addition it is imposed that the fermionic NS-R and R-NS sectors are eliminated from the physical spectrum, obtaining purely bosonic string models. Their spectrum can be determined by keeping only the string states that are left invariant by the action of the operators given in eq. (2.1), i.e.:

$$\text{Type 0A} \qquad (\mathrm{NS}-,\,\mathrm{NS}-)\otimes(\mathrm{NS}+,\,\mathrm{NS}+)\otimes(\mathrm{R}-,\,\mathrm{R}+)\otimes(\mathrm{R}+,\,\mathrm{R}-) \tag{2.6}$$

$$\text{Type 0B} \qquad (\mathrm{NS}-,\,\mathrm{NS}-)\otimes(\mathrm{NS}+,\,\mathrm{NS}+)\otimes(\mathrm{R}-,\,\mathrm{R}-)\otimes(\mathrm{R}+,\,\mathrm{R}+)\,, \tag{2.7}$$

where the signs in the various sectors refer to the values respectively taken by $(-1)^F$ and $(-1)^{\tilde{F}}$. In the (NS $-$, NS $-$) sector the lowest state is a tachyon, while the massless states live in the (NS $+$, NS $+$) sector. In the picture $(-1,-1)$ they are described by:

$$\psi^{\mu}_{-\frac{1}{2}}\tilde{\psi}^{\nu}_{-\frac{1}{2}}\,|0,\,\tilde{0},\,k\rangle_{(-1,-1)} \tag{2.8}$$

and are the same as in type II theories, namely a graviton, a dilaton and a Kalb-Ramond field. In the R-R sector, instead, we have the following massless states in the picture $(-\frac{1}{2}, -\frac{1}{2})$:

$$u_A(k)\tilde{u}_B(k)|A\rangle_{-\frac{1}{2}}|\widetilde{B}\rangle_{-\frac{1}{2}}\,. \tag{2.9}$$

Since the terms containing G and $\tilde{G}$ in the second equation in (2.1) act as the identity, the projector $P_{\mathrm{R-R}}$ imposes the existence of two kinds of R-R $(p+1)$-potentials for any value of p (C_{p+1} and $\bar{C}_{p+1}$) characterized respectively by:

$$u_A\left(\frac{1+\Gamma_{11}}{2}\right)^{A}_{\;B} = 0\,; \qquad \tilde{u}_A\left(\frac{1\pm\Gamma_{11}}{2}\right)^{A}_{\;B} = 0 \tag{2.10}$$

and by

$$u_A\left(\frac{1-\Gamma_{11}}{2}\right)^{A}_{\;B} = 0 \qquad \tilde{u}_A\left(\frac{1\mp\Gamma_{11}}{2}\right)^{A}_{\;B} = 0\,, \tag{2.11}$$

where the upper (lower) sign corresponds to 0B (0A). The doubling of the R-R potentials implies the existence of two kinds of branes that are charged with respect to both the potentials. We follow the convention of denoting by p and p' respectively branes having equal or opposite charges with respect to the two $(p+1)$ R-R potentials. The p and p'-branes are called respectively electric and magnetic branes [13]. In the case of a D3-brane the two potentials C_4 and $\tilde{C}_4$ have field strenghts F_5 and $\tilde{F}_5$ that are respectively self-dual, $F_5 = {}^*F_5$, and antiself-dual, $\tilde{F}_5 = -{}^*\tilde{F}_5$, as follows from eqs. (2.10) and (2.11). This means that, if we take the linear combinations:

$$(C_4)^{\pm} = \frac{1}{\sqrt{2}}(C_4 \pm \bar{C}_4) \tag{2.12}$$

one can see that the Hodge duality transforms each field strength into the other according to the relation ${}^*F_5^{\pm} = F_5^{\mp}$. Therefore, while the D3-brane of type IIB is naturally dyonic, in type 0B the dyonic D3-brane is constructed as a superposition of an equal number of electric and magnetic D3-branes.

The existence of two different kinds of branes in type 0 theories (the p-brane and the p'-brane) implies the one of four distinct kinds of open strings: those stretching between two p or two p'-branes (denoted by pp and $p'p'$) and those of mixed type (pp' and $p'p$). This means that the most general Chan-Paton factor λ in the expression of the open string states has the following form:

$$\lambda \equiv \begin{pmatrix} pp & pp' \\ p'p & p'p' \end{pmatrix}. \tag{2.13}$$

Open/closed string duality makes the following spin structure correspondence to hold [13]:

Interactions	Closed states	Open states	
$pp \;\; p'p'$	NS – NS	NS	
$pp \;\; p'p'$	R – R	$\mathrm{NS}(-1)^F$	(2.14)
$pp' \;\; p'p$	$\mathrm{NS} - \mathrm{NS}(-1)^F$	R	
$pp' \;\; p'p$	$\mathrm{R} - \mathrm{R}(-1)^F$	$\mathrm{R}(-1)^F$	

From this scheme, it is easy to see that the spectrum of pp and $p'p'$ strings contains only the NS and NS$(-1)^F$ sectors [13, 17] whose massless excitations are the bosons of the gauge theory. In the case of $D3$-branes one has:

$$A^\alpha \equiv \begin{pmatrix} pp & 0 \\ 0 & p'p' \end{pmatrix} \otimes \psi^\alpha_{-1/2}|0\rangle_{-1} \qquad \alpha = 0, \ldots, 3 \tag{2.15}$$

$$\phi^i \equiv \begin{pmatrix} pp & 0 \\ 0 & p'p' \end{pmatrix} \otimes \psi^i_{-1/2}|0\rangle_{-1} \qquad i = 4, \ldots, 9 \tag{2.16}$$

A^α corresponds to the gauge vector of the gauge theory, while the ϕ^i's represent six adjoint scalars. On the other hand pp' strings have only the R spectrum [13, 17] which provides fermions to the gauge theory supported by the branes. The lowest excitation of these strings are:

$$\psi^A \equiv \begin{pmatrix} 0 & pp' \\ p'p & 0 \end{pmatrix} \otimes |A\rangle_{-\frac{1}{2}}\,, \tag{2.17}$$

being $|A\rangle_{-\frac{1}{2}}$ a Majorana-Weyl spinor of the ten-dimensional Lorentz group. The interaction between two branes of the same kind is [13]:

$$\begin{aligned} Z^o_{pp} &= 2\int_0^{+\infty} \frac{d\tau}{2\tau} \mathrm{Tr}_{\mathrm{NS}}\left[e^{-2\pi\tau L_0}(-1)^{G_{bc}}\left(\frac{(-1)^G + (-1)^F}{2}\right)\right] \\ &= V_{p+1}(8\pi^2\alpha')^{-\frac{p+1}{2}} \int_0^{+\infty} \frac{d\tau}{\tau^{\frac{p+3}{2}}} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{1}{2}\left[\left(\frac{f_3(k)}{f_1(k)}\right)^8 - \left(\frac{f_4(k)}{f_1(k)}\right)^8\right], \end{aligned} \tag{2.18}$$

where o stands for open, y is the distance between the branes and G_{bc} is the ghost number operator. The interaction between a p and a p'-brane is obtained by computing the trace in the R-sector [13]:

$$\begin{aligned} Z^o_{pp'} &= 2\int_0^{+\infty} \frac{d\tau}{2\tau} \mathrm{Tr}_{\mathrm{R}}\left[e^{-2\pi\tau L_0}(-1)^{G_{bc}}\left(\frac{(-1)^G + (-1)^F}{2}\right)\right] \\ &= -V_{p+1}(8\pi^2\alpha')^{-\frac{p+1}{2}} \int_0^{+\infty} \frac{d\tau}{\tau^{\frac{p+3}{2}}} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{1}{2}\left(\frac{f_2(k)}{f_1(k)}\right)^8 . \end{aligned} \tag{2.19}$$

The explicit expressions for the functions $f_i(k)$ can be found in [11, appendix A]. Eqs. (2.18) and (2.19) can be written in a more compact form by introducing, in the trace of the free energy, the following projector:

$$P_{(-1)^{F_s}} = \frac{1 + (-1)^{F_s}}{2} \tag{2.20}$$

that, as we have mentioned above, eliminates all fermionic states from the spectrum since F_s is the space-time fermion number operator. The introduction of this operator is quite natural if one regards type 0 string theory as the orbifold type IIB$/(-1)^{F_s}$ [25]. From this point of view, the spectrum of the physical closed string states, written in eqs. (2.6) and (2.7), is made of the untwisted and twisted sectors of this orbifold. Of course the untwisted spectrum coincides with the bosonic states of type II theories. The twisted

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sector can be more easily determined using the Green-Schwarz formalism, rather than the NS-R one, due to the simple action of $(-1)^{F_s}$ on the space-time fermion coordinates S^{Aa}. Here $A = 1, 2$ and a labels the two spinor representations of the light-cone Lorentz group $SO(8)$, namely it is either an $\mathbf{8_s}$ or $\mathbf{8_c}$ index. In the twisted sector the boundary conditions on these coordinates are antiperiodic rather than periodic. Hence, the Fourier expansion for them contains half-integer fermion modes. The lowest level corresponds to a tachyon while the first one, corresponding to the massless states, is given by:

$$S^a_{-\frac{1}{2}}\tilde{S}^b_{-\frac{1}{2}}|0\rangle| \otimes |\tilde{0}\rangle \qquad \text{in Type 0B} \tag{2.21}$$

$$S^a_{-\frac{1}{2}}\tilde{S}^{\dot{b}}_{-\frac{1}{2}}|0\rangle \otimes |\tilde{0}\rangle \qquad \text{in type 0A}\,. \tag{2.22}$$

These states provide the doubling of the R-R forms previously discussed.

The spectrum of open strings can be easily obtained in the NS-R superstring formalism. Since a stack of N dyonic D-branes of type 0 contains N p-branes and N p'-branes, the massless open string states living on their world-volume are the subset of the massless open string states living on the world-volume of $2N$ Dp-branes of type II theories, that are invariant under the action of the operator $(-1)^{F_s}$. However, in order to recover the results given in eqs. (2.15), (2.16) and (2.17), we have to define a non trivial action of the space-time fermion number operator on the Chan-Paton factors, i.e. [25, 26]:

$$(-1)^{F_s}\lambda_{ij} \equiv \left(\gamma_{(-1)^{F_s}}\right)_{ih}\lambda_{hk}\left(\gamma^{-1}_{(-1)^{F_s}}\right)_{kj}\,, \tag{2.23}$$

where

$$\gamma_{(-1)^{F_s}} = \begin{pmatrix} \mathbb{I}_{N\times N} & 0 \\ 0 & -\mathbb{I}_{M\times M} \end{pmatrix} \tag{2.24}$$

and N, M denote the number of p and p'-branes respectively. The requirement of invariance for the physical states imposes the following constraints on the Chan-Paton factors:

$$\lambda^{(\mathrm{NS})} = \gamma_{(-1)^{F_s}}\lambda^{(\mathrm{NS})}\gamma^{-1}_{(-1)^{F_s}} \qquad \lambda^{(\mathrm{R})} = -\gamma_{(-1)^{F_s}}\lambda^{(\mathrm{R})}\gamma^{-1}_{(-1)^{F_s}}\,, \tag{2.25}$$

where the minus sign is due to the action of F_s on the space-time fermion $|A\rangle$. It is easy to see that the previous equations are satisfied by the matrices given in eqs. (2.15), (2.16) and (2.17). The free-energy is now written as:

$$\begin{aligned}
Z^o &= 2\int_0^{+\infty}\frac{d\tau}{2\tau}\mathrm{Tr}_{\mathrm{NS-R}}\left[e^{-2\pi\tau L_0}P_{\mathrm{GSO}}\left(\frac{1+(-1)^{F_s}}{2}\right)\right] \\
&= \frac{1}{2}\,\mathrm{Tr}[\mathbb{I}]^2 V_{p+1}(8\pi^2\alpha')^{-\frac{p+1}{2}}\int_0^{+\infty}\frac{d\tau}{\tau^{\frac{p+3}{2}}}e^{-\frac{y^2\tau}{2\pi\alpha'}}\frac{1}{2}\left[\left(\frac{f_3(k)}{f_1(k)}\right)^8 - \left(\frac{f_4(k)}{f_1(k)}\right)^8 - \left(\frac{f_2(k)}{f_1(k)}\right)^8\right] + \\
&\quad + \frac{1}{2}\,\mathrm{Tr}\left[\gamma_{(-1)^{F_s}}\right]^2 V_{p+1}(8\pi^2\alpha')^{-\frac{p+1}{2}} \times \\
&\quad \times \int_0^{+\infty}\frac{d\tau}{\tau^{\frac{p+3}{2}}}e^{-\frac{y^2\tau}{2\pi\alpha'}}\frac{1}{2}\left[\left(\frac{f_3(k)}{f_1(k)}\right)^8 - \left(\frac{f_4(k)}{f_1(k)}\right)^8 + \left(\frac{f_2(k)}{f_1(k)}\right)^8\right] \\
&\equiv Z^o_{pp} + 2Z^o_{pp'} + Z^o_{p'p'}\,.
\end{aligned} \tag{2.26}$$

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Since the traces of the Chan-Paton factors are given by

$$\mathrm{Tr}\,[\mathbb{I}]^2 = (N+M)^2 \qquad \mathrm{Tr}\left[\gamma_{(-1)^{F_s}}\right]^2 = (N-M)^2\,, \tag{2.27}$$

we can rewrite the previous equation as follows:

$$\begin{aligned} Z^o = {} & V_{p+1}(8\pi^2\alpha')^{-\frac{p+1}{2}} \int \frac{d\tau}{\tau^{\frac{p+3}{2}}} e^{-\frac{y^2\tau}{2\pi\alpha'}} \times \\ & \times \left\{ \frac{N^2+M^2}{2} \left[\left(\frac{f_3(k)}{f_1(k)}\right)^8 - \left(\frac{f_4(k)}{f_1(k)}\right)^8 \right] - MN \left(\frac{f_2(k)}{f_1(k)}\right)^8 \right\}. \end{aligned} \tag{2.28}$$

We see that eqs. (2.18) and (2.19) are obtained by putting respectively $M = 0$, $N = 1$ and $N = M = 1$ in eq. (2.28), while by taking $N = M$ we get the interaction among composite objects made of an equal number of Dp and Dp'-branes.

To conclude, let us observe that the world-volume of a dyonic D3-brane configuration (corresponding to the case $N = M$) supports a $\mathrm{U}(N)\times\mathrm{U}(N)$ gauge theory with six adjoint scalars for each gauge factor and four Weyl fermions in the bi-fundamental representation of the gauge group $(N\,,\,\bar{N})$ and $(\bar{N}\,,\,N)$. The number of bosonic degrees of freedom of the open strings attached on the N dyonic branes is $8N^2\times 2$, and coincides with the number of the fermionic ones. Therefore, the gauge theory supported by these bound states, even if non-supersymmetric, exhibits a Bose-Fermi degeneracy. From this point of view, the interaction between two dyonic branes vanishes because of a perfect cancellation between the contribution of bosonic and fermionic degrees of freedom, ensuring the stability of the configuration. The Bose-Fermi degeneracy turns out to be an essential ingredient for making the gauge/gravity correspondence to hold. We will come back on this point later.

3. Orientifold $\Omega' I_6(-1)^{F_L}$ of type $0B$

In this section we study type 0 string theory on the orientifold $\Omega' I_6(-1)^{F_L}$, where Ω' is the world-sheet parity operator, I_6 is the inversion on the six space-time coordinates labelled by $i = 4,\ldots,9$, i.e:

$$I_6 : \; (x^4,\ldots,x^9) \to (-x^4,\ldots,-x^9) \tag{3.1}$$

and F_L is the space-time fermion number operator in the left sector.

The orientifold fixed plane is, by definition, the set of points left invariant by the combined action of Ω' and I_6. In our case such a plane is located at $x^4 = \cdots = x^9 = 0$.

The action of the world-sheet parity operator Ω' in the open string sector is:

$$\Omega' \alpha_m \Omega'^{-1} = \pm e^{i\pi m}\alpha_m \qquad \Omega' \psi_r \Omega'^{-1} = \pm e^{i\pi r}\psi_r \tag{3.2}$$

for integer and half-integer r and the signs $\pm$ refer respectively to NN and DD boundary conditions. The Ω' action on the ghost and superghost oscillators is instead given by:

$$\Omega' b_n \Omega'^{-1} = e^{i\pi n} b_n \qquad \Omega' \beta_r \Omega'^{-1} = e^{i\pi r}\beta_r \tag{3.3}$$

and by analogous expressions for the c-ghost and the γ-superghost. Finally, its action on the NS and R vacua is

$$\Omega'|0\rangle_{-1} = -i|0\rangle_{-1} \tag{3.4}$$

$$\Omega'|A\rangle_{-\frac{1}{2}} = -|A\rangle_{-\frac{1}{2}} \qquad \Omega'|A\rangle_{-\frac{1}{2}} = -\Gamma^{p+1}\dots\Gamma^{9}|A\rangle_{-\frac{1}{2}}\,, \tag{3.5}$$

where the first equation in (3.5) must be used in the case of NN boundary conditions, while the second is valid for DD boundary conditions along the directions $\{p+1,\dots,9\}$.

The action of Ω' in the closed string sector is:[4]

$$\begin{array}{llll} \Omega'\alpha_n^\mu\Omega'^{-1} = \tilde{\alpha}_n^\mu & \Omega'\tilde{\alpha}_n^\mu\Omega'^{-1} = \alpha_n^\mu & \Omega'\psi_r^\mu\Omega'^{-1} = \tilde{\psi}_r^\mu & \Omega'\tilde{\psi}_r^\mu\Omega'^{-1} = \psi_r^\mu \\ \Omega' b_n\Omega'^{-1} = \tilde{b}_n & \Omega' c_n\Omega'^{-1} = \tilde{c}_n & \Omega'\beta_r\Omega'^{-1} = \tilde{\beta}_r & \Omega'\gamma_r\Omega'^{-1} = \tilde{\gamma}_r \end{array} \tag{3.6}$$

and

$$\Omega'\left(|0\rangle_{-1}\otimes|\tilde{0}\rangle_{-1}\right) = -|0\rangle_{-1}\otimes|\tilde{0}\rangle_{-1} \tag{3.7}$$

$$\Omega'\left(|A\rangle_{-\frac{1}{2}}\otimes|\tilde{B}\rangle_{\frac{1}{2}}\right) = \left(\Gamma^{11}\right)^{B}{}_{D}|D\rangle_{-\frac{1}{2}}\otimes|\tilde{A}\rangle_{-\frac{1}{2}}\,. \tag{3.8}$$

Finally, let us notice that the reason why the orientifold projector contains the term $(-1)^{F_L}$ with the space-time fermion number operator of the left sector is that, in general, the operator $\Omega' I_{2n}$ squares to unity only for n even. In fact I_{2n}^2 represents a 2π rotation in n planes and, for n odd, is equal to $(-1)^{F_s} = (-1)^{F_L+F_R}$. Therefore:

$$\left[\Omega' I_{2n}(-1)^{F_L}\right]^2 = I_{2n}^2(-1)^{F_L+F_R} = \mathbb{I} \qquad n=1,3\,, \tag{3.9}$$

where we have used the fact that $\Omega'(-1)^{F_L}\Omega'^{-1} = (-1)^{F_R}$ and $\Omega'^2 = 1$. The introduction of $(-1)^{F_L}$ is also consistent with the general property that making the orientifold projection $\Omega' I_n$ is equivalent to performing n T dualities on the unoriented theory. T-duality transforms Ω' into $\Omega' I_n(-1)^{F_L}$ where the last operator is again present only if one rotates fermions in an odd number of planes (i.e for I_2 and I_6).

3.1 Open and closed string spectrum

Let us determine the spectrum of the massless open string states attached to N D3-branes at the orientifold plane. The generic massless open string state in type 0 is given by:

$$A^\alpha \equiv \lambda_A\psi^\alpha_{-1/2}|0,k\rangle\,, \qquad \phi^i \equiv \lambda_\phi\psi^i_{-1/2}|0,k\rangle\,, \qquad \psi \equiv \lambda_\psi|s_0\dots s_4\rangle \tag{3.10}$$

being λ a $2N\times 2N$ matrix, $\alpha = 0,\dots,3$ and $i = 4,\dots,9$. By imposing the invariance under the space-time fermion number operator given in eqs. (2.20) and (2.24), we obtain for the bosonic Chan-Paton factors the diagonal structure given in eqs. (2.15) and (2.16) and for the fermionic ones the off-diagonal structure in eq. (2.17). On these states we have then to impose the orientifold projection and select only those states that are invariant under the action of $\Omega' I_6$. In the NS-sector we have:

$$\Omega' I_6\ \psi^\alpha_{-1/2}|0,k\rangle \to -\psi^\alpha_{-1/2}|0,k\rangle\,, \qquad \Omega' I_6\ \psi^i_{-1/2}|0,k\rangle \to -\psi^i_{-1/2}|0,k\rangle \tag{3.11}$$

[4] In refs. [27]–[29] a different definition of the world-sheet parity operator is used where the action of Ω' on the R-R vacuum is not as in eq. (3.8).

and therefore the invariant states satisfy the constraint:

$$\gamma_{\Omega' I_6} \lambda^T_{A,\phi} \gamma^{-1}_{\Omega' I_6} = -\lambda_{A,\phi} \,. \tag{3.12}$$

In the R-sector, instead, we have to determine how the reflection operator acts on the spinor state. By observing that a reflection in a plane corresponds to a rotation of an angle π in the plane, we can write:

$$I_6 \equiv \mathrm{e}^{\pm i\pi(S^{45}+S^{67}+S^{89})} \,, \tag{3.13}$$

being S^{ij} the zero modes of the Lorentz group generators, i.e:

$$S^{ij} = -\frac{i}{2}[\psi_0^i, \psi_0^j] \,, \qquad \sqrt{2}\psi_0^i \equiv \Gamma^i \,. \tag{3.14}$$

By introducing the operators

$$N_0 = -\frac{\Gamma^0\Gamma^1}{2} \,, \qquad N_i = -i\frac{\Gamma^{2i}\Gamma^{2i+1}}{2} \qquad \text{with } i = 1, \ldots, 4 \tag{3.15}$$

it is straightforward to verify that $N_2 \equiv S^{45}$, $N_3 \equiv S^{67}$ and $N_4 \equiv S^{89}$. In conclusion we get:

$$I_6|s_0 \ldots s_4\rangle = e^{\pm i\pi(s_2+s_3+s_4)}|s_0 \ldots s_4\rangle = \prod_{i=1}^{3}(\pm 2iN_i)|s_0 \ldots s_4\rangle = \pm\Gamma^4 \ldots \Gamma^9|s_0 \ldots s_4\rangle \,, \tag{3.16}$$

where we have taken into account that the state $|s_0 \ldots s_4\rangle$ is an eigenstate of the operator N_i with eigenvalue $s_i = \pm 1/2$. By choosing in Eq. (3.16) the minus sign we obtain the result given in ref. [30], while by choosing the plus sign we find agreement with ref. [31]. We follow the latter convention and therefore in the R-sector we write:

$$\Omega' I_6|s_0 \ldots s_4\rangle = |s_0 \ldots s_4\rangle \Longrightarrow \gamma_{\Omega' I_6} \lambda^T_{\psi} \gamma^{-1}_{\Omega' I_6} = \lambda_\psi \,. \tag{3.17}$$

In the previous equation we should also take into account the action of the operator $(-1)^{F_L}$. This is irrelevant or gives an extra minus sign in the R-sector depending whether the open string is considered respectively the right or left sector of the closed string. However, it is simple to check that this sign ambiguity is completely irrelevant in determining the spectrum of the massless states.

In the last part of this section we determine the orientifold action on the Chan-Paton factors. First we observe that the Chan-Paton factors have to be $2N \times 2N$ matrices in order to take into account their images under $\Omega' I_6$. Furthermore, following Ref. [31], they have to satisfy the constraint $\gamma_{\Omega' I_6} = \pm\gamma^T_{\Omega' I_6}$ that implies

$$\gamma_{\Omega' I_6} = \begin{pmatrix} 0 & \mathbb{I}_{N\times N} \\ \pm\mathbb{I}_{N\times N} & 0 \end{pmatrix} . \tag{3.18}$$

Substituting eq. (3.18) in eqs. (3.12) and (3.17), one gets for the bosonic and fermionic Chan-Paton factors:

$$\lambda_{A,\phi} = \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \qquad \lambda_\psi = \begin{pmatrix} 0 & B \\ \pm B^* & 0 \end{pmatrix} \tag{3.19}$$

where in the last expression we have implemented the hermiticity of the Chan-Paton factor and the matrix B can be chosen to be either symmetric or antisymmetric depending on how the sign in eq. (3.18) is chosen. The number of bosonic degrees of freedom is $8N^2$ which corresponds to one gauge boson and six real scalars transforming according to the adjoint representation of SU(N). In the fermionic sector one has $8N^2 \pm 8N$ corresponding to four Dirac fermions in the two-index symmetric (+) or antisymmetric (−) representation. Notice that the spectrum does not satisfy the Bose-Fermi degeneracy condition that holds in type 0 theory. In this case such a degeneracy is present only in the large N limit.

Moreover the spectrum of this theory has the same bosonic content as $\mathcal{N} = 4$ SYM. This is an example of planar equivalence [20] between a supersymmetric model, the $\mathcal{N} = 4$ SYM, which plays the role of the *parent* theory and a non-supersymmetric one, that is the orientifold $\Omega' I_6 (-1)^{F_L}$ of type $0B$, which is the *daughter* theory, the two being equivalent in the large N limit. In section 3.3, using string techniques, we will explicitly see that in this limit the two theories have the same β-function.

Let us consider the closed string spectrum. Since Ω' leaves invariant the metric and the dilaton, while changes sign to the Kalb-Ramond field it is easy to see that in the NS-NS sector the orientifold projection selects the following states

$$\phi\,, \quad g_{\alpha\beta}\,, \quad g_{ij}\,, \quad B_{i\alpha} \qquad \text{with} \qquad \alpha,\beta = 0,\ldots,3 \qquad i,j = 4,\ldots,9\,, \tag{3.20}$$

where ϕ, g and B are respectively the dilaton, graviton and Kalb-Ramond fields. In the R-R sector the states which are even under the orientifold projection are

$$(\mathrm{R}+,\mathrm{R}+) \to C_0\,, \quad C_{\alpha i}\,, \quad C_{0123}\,, \quad C_{\alpha\beta ij}\,, \quad C_{ijhk}\,, \tag{3.21}$$

$$(\mathrm{R}-,\mathrm{R}-) \to \bar{C}_{\alpha\beta}\,, \quad \bar{C}_{ij}\,, \quad \bar{C}_{\alpha\beta\gamma i}\,, \quad \bar{C}_{\alpha ijk}\,, \tag{3.22}$$

The previous results follow from the fact that, because of eq. (3.8), in the sector (R+, R+) where $\Gamma_{11} = 1$, Ω' leaves C_2 invariant and changes the sign of C_0 and C_4. In the sector (R−, R−) one has instead $\Gamma_{11} = -1$ and therefore Ω' leaves $\bar{C}_0$ and $\bar{C}_4$ invariant and changes the sign of $\bar{C}_2$. Notice that the R-R 5-form field strength surviving the orientifold projection is the self-dual one ($dC_4 = {}^*dC_4$), while the anti-self dual one ($d\bar{C}_4 = -\,{}^*d\bar{C}_4$) is projected out.

3.2 One-loop vacuum amplitude

In this section we compute the interaction between two stacks of N D3-branes in the orientifold $\Omega' I_6$, the action of the operator $(-1)^{F_L}$ being irrelevant in the open string calculation, as previously discussed.

The one-loop open string gets two contributions, the annulus Z_e and the Moebius strip $Z_{\Omega' I_6}$:

$$\begin{aligned} Z^o &\equiv Z^o_e + Z^o_{\Omega' I_6} \\ &= \int_0^\infty \frac{d\tau}{\tau} \mathrm{Tr}_{\mathrm{NS-R}} \left[\frac{e + \Omega' I_6}{2} \, \frac{1 + (-1)^{F_s}}{2} (-1)^{G_{bc}} \, \frac{(-1)^G + (-1)^F}{2} e^{-2\pi\tau L_0} \right]. \end{aligned} \tag{3.23}$$

The annulus contribution is equal to the one in eq. (2.28) with $M = N$ and with an extra factor 1/2 due to the orientifold projection. For $p = 3$ we get:

$$Z_e^o = N^2 \frac{V_4}{(8\pi^2\alpha')^2} \int_0^{+\infty} \frac{d\tau}{2\tau^3} \left[\left(\frac{f_3(k)}{f_1(k)}\right)^8 - \left(\frac{f_4(k)}{f_1(k)}\right)^8 - \left(\frac{f_2(k)}{f_1(k)}\right)^8 \right]. \tag{3.24}$$

In particular it vanishes because of the abstruse identity. The contribution of the Moebius strip, corresponding to the insertion of $\Omega' I_6$ in the trace, is instead non-trivial. Let us first compute, for such a term, the trace over the Chan-Paton factors. By fixing the normalization as $\langle hk|nm\rangle = \delta_{kn}\delta_{hm}$, one finds:

$$\begin{aligned} \mathrm{Tr}^{\mathrm{C.P.}} \left[\langle hk|\Omega' I_6|ij\rangle\right] &= \mathrm{Tr}\left[\gamma^{-1}_{\Omega' I_6}\gamma^{T}_{\Omega' I_6}\right] \\ \mathrm{Tr}^{\mathrm{C.P.}} \left[\langle hk|\Omega' I_6(-1)^{F_s}|ij\rangle\right] &= \mathrm{Tr}\left[\gamma^{-1}_{\Omega' I_6}\gamma^{-1}_{(-1)^{F_s}}\gamma^{T}_{\Omega' I_6}\gamma^{T}_{(-1)^{F_s}}\right]. \end{aligned} \tag{3.25}$$

Furthermore, from the explicit form of the matrices introduced in the last expression and given in eqs. (2.24) and (3.18), it is straightforward to check that

$$\mathrm{Tr}\left[\gamma^{-1}_{\Omega' I_6}\gamma^{-1}_{(-1)^{F_s}}\gamma^{T}_{\Omega' I_6}\gamma^{T}_{(-1)^{F_s}}\right] = -\mathrm{Tr}\left[\gamma^{-1}_{\Omega' I_6}\gamma^{T}_{\Omega' I_6}\right]. \tag{3.26}$$

This identity implies that the NS contribution to the free energy vanishes.

A non-vanishing contribution comes from the R sector, where the trace over the non-zero modes ($n.z.m.$) gives

$$\mathrm{Tr}_R^{n.z.m.}\left[e^{-2\pi\tau N_\psi}\Omega' I_6(-1)^G\right] = \frac{(ik)^{-2/3}}{2^4} f_2^8(ik) \tag{3.27}$$

$$\mathrm{Tr}_R^{n.z.m.}\left[e^{-2\pi\tau N_\psi}\Omega' I_6(-1)^F\right] = (ik)^{-2/3} f_1^8(ik)\,, \tag{3.28}$$

while the trace over the zero modes ($z.m.$) is given by:

$$\mathrm{Tr}_R^{z.m.}\left[\Omega' I_6(-1)^{G_0}\right] = \mathrm{Tr}_R^{z.m.}\left[(-1)^{G_0}\right] = 2^4 \tag{3.29}$$

$$\mathrm{Tr}_R^{z.m.}\left[\Omega' I_6(-1)^{F_0}\right] = \mathrm{Tr}_R^{z.m.}\left[(-1)^{F_0}\right] = 0. \tag{3.30}$$

By inserting eqs. (3.25)–(3.30) in the term with $\Omega' I_6$ in eq. (3.23), we get:

$$Z^o_{\Omega' I_6} = -\frac{V_4}{4(8\pi^2\alpha')^2}\,\mathrm{Tr}\left[\gamma^{T}_{\Omega' I_6}\gamma^{-1}_{\Omega' I_6}\right] \int_0^\infty \frac{d\tau}{\tau^3}\left(\frac{f_2(ik)}{f_1(ik)}\right)^8, \tag{3.31}$$

where we should use that $\mathrm{Tr}[\gamma^{T}_{\Omega' I_6}\gamma^{-1}_{\Omega' I_6}] = \pm 2N$. Notice that, because of the Moebius strip contribution, the interaction between two D3 branes in this orientifold, is non vanishing. We would like to point out that the T-dual versions with D9-branes of eqs. (3.24) and (3.31) appear in ref. [17].

The previous equation, together with the third term of eq. (3.24), gives the total fermionic contribution to the free-energy which at the massless level reduces to:

$$Z^o(\text{femionic massless}) = -(8N^2 \pm 8N)\frac{V_4}{(8\pi^2\alpha')^2}\int_0^\infty \frac{d\tau}{\tau^3}. \tag{3.32}$$

As usual, the factor $(8N^2 \pm 8N)$ in front of the previous expression counts the number of the fermionic degrees of freedom of the world-volume gauge theory, which indeed agrees with the counting of the previous subsection. As already noticed, we do not have the same number of bosonic and fermionic degrees of freedom propagating in the loop and, in particular, the additional $\pm 8N$ fermionic term comes from the Moebius strip, which therefore is responsable of spoiling the Bose-Fermi degeneracy of the theory [26]. This contribution is subleading in the large N limit.

Notice that eq. (3.31), apart from the Chan-Paton factors and the substitution $k \to ik$, is 1/2 of the free energy describing, in the R-sector, the interaction between two D3-branes in type IIB string theory. It is, by the way, also equal, with the previous substitutions, to the correspondent expression in type 0 theory, given in eq. (2.19).

3.3 One-loop vacuum amplitude with an external field

Let us consider, in the open channel, the interaction between a D3-brane dressed with a constant SU(N) gauge field and a stack of N undressed D3-branes. The gauge field is chosen to have only the entries $\hat{F}_{01} = 2\pi\alpha' \hat{f}$ and $\hat{F}_{23} = 2\pi\alpha'\hat{g}$ different from zero. The presence of the external field modifies eq. (3.31) as follows:

$$Z^o(F)_{\Omega' I_6} = \frac{2\,\mathrm{Tr}\left[\gamma^T_{\Omega' I_6}\gamma^{-1}_{\Omega' I_6}\right]}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta+\hat{F})} \int_0^\infty \frac{d\tau}{\tau} \mathrm{e}^{-\frac{y^2\tau}{2\pi\alpha'}} \sin\pi\nu_f \sin\pi\nu_g \times \\ \times \frac{f_2^4(ik)\Theta_2\left(i\nu_f\tau|i\tau+1/2\right)\Theta_2\left(i\nu_g\tau|i\tau+1/2\right)}{f_1^4(ik)\Theta_1\left(i\nu_f\tau|i\tau+1/2\right)\Theta_1\left(i\nu_g\tau|i\tau+1/2\right)}, \tag{3.33}$$

where [11]:

$$\nu_f = \frac{1}{2\pi i}\log\frac{1+2\pi\alpha'\hat{f}}{1-2\pi\alpha'\hat{f}}, \qquad \nu_g = \frac{1}{2\pi i}\log\frac{1-2\pi i\alpha'\hat{g}}{1+2\pi i\alpha'\hat{g}}. \tag{3.34}$$

Notice that in the previous expression we should have put $y = 0$ because all branes are located at the orientifold point. However we keep $y \neq 0$ because it provides a natural infrared cutoff. Eq. (3.33) describes the Moebius strip with the boundary on the dressed brane. The trace over the Chan-Paton factors gives ± 2 counting the dressed brane and its image under $\Omega' I_6$. The overall factor 2, instead, is a consequence of the fact that in the trace we have to sum over two different but equivalent open string configurations: the first one in which only the end-point of the string parametrized by the world-sheet coordinate $\sigma = 0$ is charged under the gauge group, and the other one in which the gauge charge is turned instead on the other end-point at $\sigma = \pi$.

We are now going to compute the field theory limit of eq. (3.33) which gives the corresponding Euler-Heisenberg effective action and the threshold corrections to the running coupling constant.

The field theory limit can be performed by taking $\alpha' \to 0$ and $\tau \to \infty$ in such a way that the Schwinger parameter of the field theory $\sigma = 2\pi\alpha'\tau$ is kept fixed. Using in this limit the equations (A.7)–(A.10) given in the appendix, from eq. (3.33) we get:

$$Z^F \simeq \mp\frac{16}{(4\,\pi)^2}\int d^4x \int_0^\infty \frac{d\sigma}{\sigma}\mathrm{e}^{-\frac{y^2}{(2\pi\alpha')^2}\,\sigma}\;\hat{f}\,\hat{g}\;\frac{\cos(\sigma\hat{f})\cosh(\sigma\hat{g})}{\sin(\sigma\hat{f})\sinh(\sigma\hat{g})}. \tag{3.35}$$

Finally, expanding eq. (3.35) up to the quadratic order in the gauge fields yields:

$$Z^F(F^2) \simeq \mp\frac{16}{(4\,\pi)^2}\int d^4x \int_{1/\Lambda^2}^\infty \frac{d\sigma}{\sigma^3}\mathrm{e}^{-\mu^2\sigma}\left[1-\frac{1}{3}\sigma^2(\hat{f}^2\,-\,\hat{g}^2)\right], \tag{3.36}$$

where Λ is an UV cut-off and $\mu = \frac{y}{2\pi\alpha'}$ is an IR one. By using

$$\hat{f}^2 - \hat{g}^2 = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \,, \tag{3.37}$$

one gets the following expression for the running coupling constant:

$$\frac{1}{g^2_{\rm YM}(\mu)} = \frac{1}{g^2_{\rm YM}(\Lambda)} \mp \frac{1}{3\pi^2} \log \frac{\mu^2}{\Lambda^2} \,, \tag{3.38}$$

where the tree diagram contribution has been introduced by hand. Finally from eq. (3.38) one reads the expected β-function

$$\beta(g_{YM}) = \pm \frac{g^3_{YM}}{(4\pi)^2} \frac{16}{3} \,. \tag{3.39}$$

As already observed, in the planar limit $N \to \infty$ with $\lambda = g^2_{\rm YM} N$ fixed, the ratio $\beta(g_{\rm YM})/g_{\rm YM}$ reduces to zero and coincides with the one of its parent theory $\mathcal{N} = 4$ SYM.

In the last part of this subsection, in order to explore the validity of gauge/gravity correspondence in this non-supersymmetric model, we evaluate the threshold corrections to the running coupling constant.

The starting point is again eq. (3.33) that we now expand up to the quadratic order in the gauge fields without performing any field theory limit (more details on the calculation can be found in the appendix) obtaining ($k = e^{-\pi\tau}$):

$$\frac{1}{g^2_{\rm YM}} = \pm \frac{1}{(4\pi)^2} \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \left(\frac{f_2(ik)}{f_1(ik)} \right)^8 \left[\frac{1}{3\tau^2} + k \frac{\partial}{\partial k} \log f_2^4(ik) \right] . \tag{3.40}$$

Moreover by performing the field theory limit and keeping the product $\sigma = 2\pi\alpha'\tau$ fixed, we get the same expression of the running coupling constant obtained from the Euler-Heisenberg action given in eq. (3.38).

It is also interesting to write the interaction given by eq. (3.33) in the closed string channel by performing the modular transformation $\tau = 1/4t$, as shown in Appendix:

$$\begin{aligned} Z^c(F)_{\Omega' I_6} = &\mp \frac{1}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta + \hat{F})} \int_0^\infty \frac{dt}{t^3} \mathrm{e}^{-\frac{y^2}{8\pi\alpha' t}} \sin\pi\nu_f \sin\pi\nu_g \times \\ &\times \frac{f_2^4(iq)\Theta_2\left(\frac{\nu_f}{2}|it+\frac{1}{2}\right)\Theta_2\left(\frac{\nu_g}{2}|it+\frac{1}{2}\right)}{f_1^4(iq)\Theta_1\left(\frac{\nu_f}{2}|it+\frac{1}{2}\right)\Theta_1\left(\frac{\nu_g}{2}|it+\frac{1}{2}\right)} . \end{aligned} \tag{3.41}$$

Expanding the previous equation up to the second order in the external field gives (with $q = e^{-\pi t}$):

$$\frac{1}{g^2_{\rm YM}} = \pm \frac{1}{(4\pi)^2} \int_0^\infty \frac{dt}{4t^3} e^{-\frac{y^2}{8\pi\alpha' t}} \left(\frac{f_2(iq)}{f_1(iq)} \right)^8 \left[\frac{4}{3} + \frac{1}{\pi} \partial_t \log f_2^4(iq) \right] . \tag{3.42}$$

Under the inverse modular transformation $t = 1/4\tau$ this equation perfectly reproduces the expression obtained in the open channel (eq. (3.40)), as one can easily check by using eqs. (A.31).

The field theory limit of the previous expression, realized as $t \to \infty$ and $\alpha' \to 0$ with $s = 2\pi\alpha' t$ fixed, gives a vanishing contribution to eq. (3.42). One could be led to conclude that the gauge/gravity correspondence does not work in this non-supersymmetric model. However, in the planar limit ($N \to \infty$ and $g^2_{\rm YM}N$ fixed) the theory has a vanishing β-function, as noticed after eq. (3.39). Therefore one can conclude that *the gauge/gravity correspondence holds in the large N limit*, where the Moebius strip contribution is suppressed and *the gauge theory recovers the Bose-Fermi degeneracy in its spectrum.*

4. Orbifolds of previous orientifolds

In this section we consider some orbifolds of the previous orientifold theory and, within this framework, we analyze the world-volume gauge theory living on N fractional branes. Those are branes fixed at the orbifold singularity and having a non conformal gauge theory on their world-volume. The open strings attached to them have Chan-Paton factors transforming according to irreducible representations of the orbifold group and this property makes them the most elementary solitonic objects in the theory. We start by discussing the gauge theory living on N fractional D3-branes in the orbifold $C^2/\mathbb{Z}_2$. In the planar limit this theory, which is not supersymmetric, shows some interesting common features with $\mathcal{N} = 2$ super Yang-Mills.

Then we will turn to the more interesting case of the orbifold $C^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Here the gauge theory living on N fractional branes is the one recently discussed by Armoni-Shifman-Veneziano, that for $N = 3$ reduces to QCD with one flavour.

4.1 Orbifold $C^2/\mathbb{Z}_2$

The $\mathbb{Z}_2$ group is characterized by two elements $(1, h)$, being 1 the identity element and h such that $h^2 = 1$. We choose $\mathbb{Z}_2$ to act on the coordinates x^m with $m = 6, 7, 8, 9$. Therefore, introducing the complex combinations $\vec{z} = (z^1, z^2)$, where $z^1 = x^6 + ix^7$, $z^2 = x^8 + ix^9$, the non trivial element of the orbifold group acts on them as follows:

$$h : (z_1\,,\, z_2) \to (-z_1\,,\, -z_2)\,. \tag{4.1}$$

The gauge theory living on the fractional D3-branes is the $\mathbb{Z}_2$ invariant subsector of the open string spectrum introduced in section 3.1. In order to construct it explicitly one also needs to know the action of the orbifold on the spinors states:

$$h : |s_0 = \frac{1}{2}, s_1\, s_2\, s_3\, s_4\rangle \to e^{i\pi(s_3+s_4)}|s_0 = \frac{1}{2}, s_1\, s_2\, s_3\, s_4\rangle\,, \tag{4.2}$$

where $s_i = \pm 1/2$ and s_0 has been fixed to 1/2 by the mass-shell condition. Using eqs. (4.1) and (4.2) it is easy to see that the spectrum contains one SU(N) gauge field, two real scalars in the adjoint representation of the gauge group and two Dirac fermions in the two-index symmetric (or antisymmetric) representation. Notice that the spectrum has a *common sector* [20] with $\mathcal{N} = 2$ SYM, namely the bosonic one. However, because of the fermionic contributions which are different, the one-loop β-function of our theory contains a subleading correction in $1/N$ with respect to $\mathcal{N}=2$ β-function:

$$\beta(g_{YM}) = \frac{g_{YM}^3}{(4\pi)^2}\left[-\frac{11}{3}N + 2\frac{N}{6} + 2\frac{4}{3}\frac{N\pm 2}{2}\right] = \frac{g_{YM}^3 N}{(4\pi)^2}\left[-2 \pm \frac{8}{3N}\right]. \tag{4.3}$$

In the large N limit the subleading term in $1/N$ is suppressed and the two β-functions coincide. This circumstance signals the existence of a planar equivalence between the two theories at one-loop and suggests the possibility of an extension of such equivalence at all perturbative orders.

The free energy, which describes the interaction between a stack of N undressed D3 branes and a dressed one, is given by:

$$\begin{aligned} Z &= \int_0^\infty \frac{d\tau}{\tau}\mathrm{Tr}_{\mathrm{NS-R}}\left[\left(\frac{1+h}{2}\right)\left(\frac{e+\Omega' I_6}{2}\right)\left(\frac{1+(-1)^{F_s}}{2}\right)(-1)^{G_{bc}}\left(\frac{(-1)^G+(-1)^F}{2}\right)e^{-2\pi\tau L_0}\right] \\ &\equiv Z_e^o + Z_{\Omega' I_6}^o + Z_{he}^o + Z_{h\Omega' I_6}^o\,, \end{aligned} \tag{4.4}$$

where the trace over the Chan-Paton factors has been understood and we have used the following notation:

$$\begin{aligned} Z_e^o &\equiv \frac{\left(Z_e^o + Z_{e(-1)^{F_s}}^o\right)}{2} & \qquad Z_{he}^o &\equiv \frac{\left(Z_{he}^o + Z_{he(-1)^{F_s}}^o\right)}{2} \\ Z_{\Omega' I_6}^o &\equiv \frac{\left(Z_{\Omega' I_6}^o + Z_{\Omega' I_6(-1)^{F_s}}^o\right)}{2} & \qquad Z_{h\Omega' I_6}^o &\equiv \frac{\left(Z_{h\Omega' I_6}^o + Z_{h\Omega' I_6(-1)^{F_s}}^o\right)}{2}\,. \end{aligned} \tag{4.5}$$

However notice that the term $(-1)^{F_s}$ gives a non vanishing contribution to the trace on the Chan-Paton factors, only if it appears together with the projector $\Omega' I_6$, namely in the terms appearing in the second line of eq. (4.5).

The first two terms of eq. (4.4) are those that we have already computed in the previous section (apart from an additional factor $\frac{1}{2}$ coming from the orbifold projection). Here we need just to evaluate the last two terms. The third one turns out to be

$$\begin{aligned} Z_{he}^o =& \frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty \frac{d\tau}{\tau}\mathrm{e}^{-\frac{y^2\tau}{2\pi\alpha'}}\left[\frac{4\,\sin\pi\nu_f\sin\pi\nu_g}{\Theta_2^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)}\right]\times \\ &\times\left[\Theta_3^2(0|i\tau)\Theta_4(i\nu_f\tau|i\tau)\Theta_4(i\nu_g\tau|i\tau) - \Theta_4^2(0|i\tau)\Theta_3(i\nu_f\tau|i\tau)\Theta_3(i\nu_g\tau|i\tau)\right] + \\ &+ \frac{iN}{32\pi^2}\int d^4x\, F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\int_0^\infty \frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}} \end{aligned} \tag{4.6}$$

and is equal to the one appearing in the pure orbifold calculation [11]. To get the previous equation we have used:

$$\mathrm{Tr}\langle ij|e|nm\rangle = \delta_{jj}\delta_{mm} = 4N \qquad \mathrm{Tr}\langle ij|(-1)^{F_S}|nm\rangle = 0\,, \tag{4.7}$$

where the indices $i,m = 1,\ldots,N,\ N+3,\ldots,2N+2$ enumerate respectively the stack of N undressed branes and their images, while the indices $j,n = N+1, N+2$ indicate the dressed brane and its image.

Analogously the last term in eq. (4.4) can be obtained from eq. (4.6) with the substitution $k \to ik$. As noticed after [11, eq. (2.5)], in the twisted sector, only the NS and $NS(-1)^F$ and $R(-1)^F$ sectors contribute to the interaction . However, the presence of the type 0B projector $\frac{1+(-1)^{F_s}}{2}$ makes the NS and $NS(-1)^F$ contributions to vanish because of eqs. (3.25) and (3.26). Thus the only twisted sector which gives a non vanishing contribution to the interaction is the $R(-1)^F$ which is equal to

$$Z^o_{h\Omega' I_6} = \pm \frac{2i}{32\pi^2} \int d^4x \, F^a_{\alpha\beta} \tilde{F}^{a\,\alpha\beta} \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \, . \tag{4.8}$$

The overall factor 2 again takes in account the two inequivalent configurations that we have to consider in evaluating the trace, as discussed after eq. (3.34). Therefore the coefficient of the kinetic term of the gauge field in this theory comes only from the second and third term of eq. (4.4) and it turns out to be:

$$\frac{1}{g^2_{YM}} = \frac{1}{16\pi^2} \left[2N \mp \frac{8}{3} \right] \log \frac{\mu^2}{\Lambda^2} \tag{4.9}$$

consistently with our previous calculation in eq. (4.3). Moreover from eqs. (4.6) and (4.8), following the same procedure as in ref. [11], one can read also the θ angle which turns out to be

$$\theta_{YM} = 2\theta(N \pm 2) \, , \tag{4.10}$$

where θ is the phase of the complex cut-off $\Lambda e^{-i\theta}$.

The gauge theory here obtained shares some common features with $\mathcal{N} = 2$ SYM. As previously noticed, the running coupling constant and the β-function of this theory, in the large N limit, reproduce those of $\mathcal{N} = 2$ SYM. Moreover also the θ angle in eq. (4.10) in the large N limit reduces to the one of $\mathcal{N} = 2$ SYM, implying that, in the planar limit, the two theories are very close to each other. This connection appears as the natural extension to the case $\mathcal{N} = 2$ of the Armoni, Shifman and Veneziano planar equivalence for $\mathcal{N} = 1$, in which the *parent* theory is the $\mathcal{N} = 2$ SYM and the *daughter* theory is the world-volume theory of N fractional branes of our orbifold.

Finally, it is useful to rewrite the previous expressions in the closed string channel where, because of the open/closed string duality, these amplitudes correspond to the tree level exchange diagram between a stack of N undressed branes and one brane dressed with an SU(N) gauge field. In particular, by using eqs. (A.20) and (A.30) and the well-known modular transformation properties of the Θ-functions, we can rewrite eqs. (4.6), (4.8) and $Z^o_{\Omega' I_6}$ in the closed string channel. The other terms in the free energy are irrelevant in the forthcoming discussion because they are vanishing in the field theory limit. From the annulus we obtain:

$$\begin{aligned} Z^c_{he} = & \frac{N}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta + \hat{F})} \int_0^\infty \frac{dt}{t} \mathrm{e}^{-\frac{y^2}{2\pi\alpha' t}} \left[\frac{4 \sin\pi\nu_f \sin\pi\nu_g}{\Theta_4^2(0|it)\Theta_1(\nu_f|it)\Theta_1(\nu_g|it)} \right] \times \\ & \times \left[\Theta_2^2(0|it)\Theta_3(\nu_f|it)\Theta_3(\nu_g|it) - \Theta_3^2(0|it)\Theta_2(\nu_f|it)\Theta_2(\nu_g|it) \right] + \\ & + \frac{iN}{32\pi^2} \int d^4x \, F^a_{\alpha\beta} \tilde{F}^{a\,\alpha\beta} \int_0^\infty \frac{dt}{t} e^{-\frac{y^2}{2\pi\alpha' t}} \end{aligned} \tag{4.11}$$

while from the Moebius strip:

$$
\begin{aligned}
Z^c_{\Omega' I_6}(F) &= \mp\frac{1}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty \frac{dt}{4t^3}\mathrm{e}^{-\frac{y^2}{8\pi\alpha' t}}\sin\pi\nu_f\sin\pi\nu_g\times\\
&\quad\times\frac{f_2^4(iq)\Theta_2\left(\frac{\nu_f}{2}|it+\frac{1}{2}\right)\Theta_2\left(\frac{\nu_g}{2}|it+\frac{1}{2}\right)}{f_1^4(iq)\Theta_1\left(\frac{\nu_f}{2}|it+\frac{1}{2}\right)\Theta_1\left(\frac{\nu_g}{2}|it+\frac{1}{2}\right)} \qquad (4.12)\\
Z^c_{h\Omega' I_6}(F) &= \pm\frac{2i}{32\pi^2}\int d^4x\, F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\int_0^\infty\frac{dt}{t}e^{-\frac{y^2}{8\pi\alpha' t}}\,. \qquad (4.13)
\end{aligned}
$$

By expanding the previous expressions up to quadratic terms in the external field and isolating only the terms depending on the gauge field, we have from the annulus:

$$
\begin{aligned}
Z^c_h(F)\to&\left[-\frac{1}{4}\int d^4x F^a_{\alpha\beta}F^{a\,\alpha\beta}\right]\left\{-\frac{N}{8\pi^2}\int_0^\infty\frac{dt}{t}\mathrm{e}^{-\frac{y^2}{2\pi\alpha' t}}\right\}+\\
&+iN\left[\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\int_0^\infty\frac{dt}{t}\mathrm{e}^{-\frac{y^2}{2\pi\alpha' t}} \qquad (4.14)
\end{aligned}
$$

and from the Moebius strip:

$$
Z^c_{\Omega' I_6}(F)\to\left[-\frac{1}{4}\int d^4x F^a_{\alpha\beta}F^{a\,\alpha\beta}\right]\left\{\pm\frac{1}{(4\pi)^2}\int_0^\infty\frac{dt}{8t^3}e^{-\frac{y^2}{8\pi\alpha' t}}\left(\frac{f_2(iq)}{f_1(iq)}\right)^8\left[\frac{4}{3}+\frac{1}{\pi}\partial_t\log f_2^4(iq)\right]\right\}. \qquad (4.15)
$$

Eqs. (4.13) and (4.14) are exact at string level even if they receive contribution only from the massless closed string states. For $y=0$ both of them are left invariant under open/closed string duality and for this reason one is able to obtain, from the closed channel, the planar contribution to the β-function and the complete expression of the θ-angle.

Eq. (4.15), which in the open channel gives the subleading behaviour in the large N limit of the β-function, receives, instead, contributions from all the string states. The massless pole in the open channel is not left invariant under open/closed string duality and by performing the field theory limit on such expression, as explained in the last section, we obtain a vanishing result.

In conclusion we can assert that the gauge/gravity correspondence certainly holds in the planar limit where a Bose-Fermi degeneracy is recovered and the theory resembles to a supersymmetric theory. However, some non planar information can be still obtained from the closed channel as the example of the θ-angle has showed.

4.2 Orbifold $C^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ — A simple string realization of the Armoni, Shifman and Veneziano model

Let us consider now the orbifold $C^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$, which contains four elements $\{1,h_1,h_2,h_3\}$ acting on the three complex coordinates

$$
z_1=x_4+ix_5 \qquad z_2=x_6+ix_7 \qquad z_3=x_8+ix_9 \qquad (4.16)
$$

as follows:

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R_{h_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$
$$R_{h_2} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad R_{h_3} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.17}$$

and on the spinors as:

$$\begin{aligned} h_1|s_1\, s_2\, s_3\, s_4\rangle &= e^{i\pi(s_3+s_4)}|s_1\, s_2\, s_3\, s_4\rangle \\ h_2|s_1\, s_2\, s_3\, s_4\rangle &= e^{i\pi(s_2+s_4)}|s_1\, s_2\, s_3\, s_4\rangle \\ h_3|s_1\, s_2\, s_4\, s_4\rangle &= -e^{i\pi(s_2+s_3)}|s_1\, s_2\, s_3\, s_4\rangle\,, \end{aligned} \tag{4.18}$$

where the sign in front of the last equation is required by the group properties.

The states left invariant are one gauge vector and one Dirac fermion in the two-index symmetric (or anti-symmetric) representation of the gauge group. Also in this case the theory has a common bosonic sector with a supersymmetric model, that is $\mathcal{N} = 1$ SYM. It is simple to check that the β-function for this theory is, at one-loop

$$\beta(g_{YM}) = \frac{g_{YM}^3}{(4\pi)^2}\left[-\frac{11}{3}N + \frac{4}{3}\frac{N \pm 2}{2}\right] = \frac{g_{YM}^3 N}{(4\pi)^2}\left[-3 \pm \frac{4}{3N}\right] \tag{4.19}$$

which differs from the one of $\mathcal{N} = 1$ SYM because of the subleading term in $1/N$.

Notice that for $N = 3$ the two-index antisymmetric representation is equal to the fundamental one. Therefore, with the antisymmetric choice, *the world-volume gauge theory living on a stack of N fractional branes in the orbifold $C^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ of the orientifold type $0B/\ \Omega' I_6(-1)^{F_L}$, for $N = 3$ is nothing but one flavour QCD*. This is an alternative and simpler stringy realization of the Armoni-Shifman-Veneziano model. In [20] and reference therein, the same gauge theory is realized, in the framework of type $0A$ theory, by considering a stack of N D4 branes on top of an orientifold $O4$ plane, suspended between orthogonal NS 5 branes. It would be interesting to exploit the relation between the two models, which should be connected through a simple T-duality.

Besides the stringy realization, the gauge theory we end with is related by planar equivalence to $\mathcal{N} = 1$ SYM. In the language of Armoni, Shifmann and Veneziano, the symmetric $(+)$ and antisymmetric $(-)$ choices correspond to the S and A orientifold theories of $\mathcal{N} = 1$ SYM. This opens the way to a very interesting extension of many predictions of supersymmetric parent theory to the non supersymmetric daughter theory, which holds in the large N limit [20].

As discussed in refs. [32, 33], the orbifold $C^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ can be seen as obtained by three copies of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ where the i-th $\mathbb{Z}_2$ contains the elements $(1, h_i)$ $(i = 1, \ldots 3)$.

In particular we consider the interaction between a stack of N_I ($I = 1, \ldots, 4$) branes of type I and a D3-fractional brane of type $I = 1$ dressed with an SU(N) gauge field. In this case the interaction turns out to be given by the sum of eight terms:

$$
\begin{aligned}
Z &= \int_0^\infty \frac{d\tau}{\tau}\, \mathrm{Tr}_{\mathrm{NS-R}}\left[\left(\frac{1+h_1+h_2+h_3}{4}\right)\left(\frac{e+\Omega' I_6}{2}\right)\left(\frac{1+(-1)^{F_s}}{2}\right)\times\right.\\
&\qquad\qquad \left.\times\,(-1)^{G_{bc}}\left(\frac{(-1)^{G_{\beta\gamma}}+(-1)^F}{2}\right)e^{-2\pi\tau L_0}\right]\\
&\equiv Z^o_e + Z^o_{\Omega' I_6} + \sum_{i=1}^{3}\left[Z^o_{h_i e} + Z^o_{h_i \Omega' I_6}\right]. \qquad (4.20)
\end{aligned}
$$

Here the first two terms are the same as the ones of the previous orbifolds except for a further factor 1/2 due to the orbifold projection. The terms $Z^o_{h_i e}$ turn out to be

$$
\begin{aligned}
Z^o_{h_i} &= \frac{f_i(N)}{2\,(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty \frac{d\tau}{\tau}\mathrm{e}^{-\frac{y^2\tau}{2\pi\alpha'}}\frac{4\,\sin\pi\nu_f\sin\pi\nu_g}{\Theta_2^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)}\times\\
&\times\left\{\Theta_3^2(0|i\tau)\Theta_4(i\nu_f\tau|i\tau)\Theta_4(i\nu_g\tau|i\tau)-\Theta_4^2(0|i\tau)\Theta_3(i\nu_f\tau|i\tau)\Theta_3(i\nu_g\tau|i\tau)\right\}+\\
&+\frac{i\,f_i(N)}{64\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\int_0^\infty\frac{d\tau}{\tau}\mathrm{e}^{-\frac{y^2\tau}{2\pi\alpha'}}, \qquad (4.21)
\end{aligned}
$$

where the functions $f_i(N)$ depend on the number of different kinds of fractional branes N_I and are given by [32, 33]:

$$
\begin{aligned}
f_1(N_I) &= N_1 + N_2 - N_3 - N_4\\
f_2(N_I) &= N_1 - N_2 + N_3 - N_4\\
f_3(N_I) &= N_1 - N_2 - N_3 + N_4\,. \qquad (4.22)
\end{aligned}
$$

As in the previous orbifold case, all the bosonic terms of $Z_{h_i\Omega' I_6}$ vanish because of the contribution to the trace of the projector $\frac{1+(-1)^{F_s}}{2}$, while the $R(-1)^F$ sector gives

$$
Z^o_{h_i\Omega' I_6} = \pm\frac{2i}{64\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\int\frac{d\tau}{\tau}\mathrm{e}^{-\frac{y^2\tau}{2\pi\alpha'}}\,. \qquad (4.23)
$$

By extracting the coefficient of the gauge kinetic term from the field theory limit of the interaction in eq. (4.20) and specializing to the case $N_1 = N$, $N_2 = N_3 = N_4 = 0$ we get:

$$
\frac{1}{g^2_{YM}} = \frac{1}{16\pi^2}\left[3N \mp \frac{4}{3}\right]\log\frac{\mu^2}{\Lambda^2}\,, \qquad (4.24)
$$

while the theta angle turns out to be

$$
\theta_{\mathrm{YM}} = (N \pm 2)\theta\,. \qquad (4.25)
$$

We can repeat the same analysis in the closed string channel by transforming under open/closed string duality eq. (4.20) and performing all the steps explained in the latter section. However, being eqs. (4.21), (4.23) and $Z^o_{\Omega' I_6}$ coincident, apart from an overall factor, with eqs. (4.6) and (4.8) we get the same conclusions as we did in the last subsection. In the closed string channel one is able to capture, in the field theory limit, only the planar contribution to the β-funtion and the complete expression for the θ-angle. Gauge/gravity correspondence in these non-supersymmetric models holds only in the large N limit even if some non planar results are still present in the closed channel.

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A. Some useful properties of Θ functions

In this appendix we give a detailed derivation of eqs. (3.40) and (3.42).

The Θ-functions which are the solutions of the heat equation:

$$\frac{\partial}{\partial\tau}\Theta\left(\nu|i\tau\right) = \frac{1}{4\pi}\partial_\nu^2\Theta(\nu|i\tau) \tag{A.1}$$

are given by:

$$\begin{aligned}
\Theta_1(\nu|it) &\equiv \Theta_{11}(\nu,|it) = -2q^{\frac{1}{4}}\sin\pi\nu\prod_{n=1}^{\infty}\left[(1-q^{2n})(1-e^{2i\pi\nu}q^{2n})(1-e^{-2i\pi\nu}q^{2n})\right] \\
\Theta_2(\nu|it) &\equiv \Theta_{10}(\nu,|it) = 2q^{\frac{1}{4}}\cos\pi\nu\prod_{n=1}^{\infty}\left[(1-q^{2n})(1+e^{2i\pi\nu}q^{2n})(1+e^{-2i\pi\nu}q^{2n})\right] \\
\Theta_3(\nu,|it) &\equiv \Theta_{00}(\nu,|it) = \prod_{n=1}^{\infty}\left[(1-q^{2n})(1+e^{2i\pi\nu}q^{(2n-1)})(1+e^{-2i\pi\nu}q^{2n-1})\right] \\
\Theta_4(\nu,|it) &\equiv \Theta_{01}(\nu,|it) = \prod_{n=1}^{\infty}\left[(1-q^{2n})(1-e^{2i\pi\nu}q^{2n-1})(1-e^{-2i\pi\nu}q^{2n-1})\right]
\end{aligned} \tag{A.2}$$

with $q = e^{-\pi t}$. It is also useful to give an alternative representation of the Θ-functions:

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}(\nu|t) = \sum_{n=-\infty}^{\infty} e^{2\pi i\left[\frac{1}{2}(n+\frac{a}{2})^2 t + (n+\frac{a}{2})(\nu+\frac{b}{2})\right]}, \tag{A.3}$$

where a, b are rational numbers. It is easy to show that

$$\Theta\begin{bmatrix} 1 \\ 1 \end{bmatrix}(\nu|t) = -i\sum_{n=-\infty}^{\infty}(-1)^n e^{i\pi t(n-\frac{1}{2})^2}e^{i\pi\nu(2n-1)} \equiv \Theta_1\left(\nu|t\right) \tag{A.4}$$

and

$$\Theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(\nu|t) = \sum_{n=-\infty}^{\infty} e^{i\pi t(n-\frac{1}{2})^2}e^{i\pi\nu(2n-1)} = \Theta_2\left(\nu|t\right) \equiv -\Theta_1\left(\nu+\frac{1}{2}|t\right). \tag{A.5}$$

From the definition in eq. (A.3) it is easy to derive the following identity:

$$\Theta\begin{bmatrix} a \\ b \end{bmatrix}\left(\nu+\frac{\epsilon_1}{2}t+\frac{\epsilon_2}{2}|t\right) = e^{-\frac{i\pi t\epsilon_1^2}{4}}e^{-\frac{i\pi\epsilon_1}{2}(2\nu+b)}e^{-\frac{i\pi\epsilon_1\epsilon_2}{2}}\Theta\begin{bmatrix} a+\epsilon_1 \\ b+\epsilon_2 \end{bmatrix}(\nu|t). \tag{A.6}$$

In order to get the Euler-Heisenberg action in eq. (3.35) we need to use the following expressions which hold for $\tau \to \infty$ and $\alpha' \to 0$:

$$\Theta_1\left(i\nu\tau|i\tau+\frac{1}{2}\right) \to -2i(ik)^{1/4}\sinh\pi\nu\tau, , \quad \Theta_2\left(i\nu\tau|i\tau+\frac{1}{2}\right) \to 2(ik)^{1/4}\cosh\pi\nu\tau \tag{A.7}$$

$$\nu_f \to -2\alpha' i\hat{f}\,, \qquad \nu_g \to -2\alpha'\hat{g} \tag{A.8}$$

and

$$f_1(ik) \to (ik)^{1/12} \quad , \quad f_2(ik) \to \sqrt{2}(ik)^{1/12} \tag{A.9}$$

which, together with

$$\sqrt{-\det(\eta+\hat{F})}\sin\pi\nu_f\sin\pi\nu_g = i(2\pi\alpha')^2\hat{f}\hat{g}\,, \tag{A.10}$$

leads to Eq. (3.35).

In order to derive eq. (3.40) we need the following expansions of the Θ-functions up to the quadratic order in the gauge fields:

$$\begin{aligned}\Theta_n\left[i\nu_f\tau|i\tau+\frac{1}{2}\right] &\simeq \Theta_n\left[0|i\tau+\frac{1}{2}\right]+2\frac{\tau^2}{\pi}\partial_\tau\Theta_n\left[0|i\tau+\frac{1}{2}\right]f^2\\ &= f_1(ik)f_n^2(ik)\left[1+2\frac{\tau^2}{\pi}\partial_\tau\log[f_1(ik)f_n^2(ik)]\,f^2\right]\end{aligned} \tag{A.11}$$

for $n = 2, 3, 4$ and

$$\frac{\sin\pi\nu_f}{\Theta_1\left[i\nu_f\tau|i\tau+1/2\right]} \simeq \frac{i}{2\tau f_1^3(ik)}\left[1+\left(\frac{1}{6}+\tau^2 k\frac{\partial}{\partial k}\log f_1^2(ik)\right)f^2\ldots\right] \tag{A.12}$$

for Θ_1. Inserting them in eq. (3.33) we get ($k = e^{-\pi\tau}$):

$$\begin{aligned}Z^F &\simeq \pm 4\frac{1}{(8\pi^2\alpha')^2}\int d^4x\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\left[1-\frac{1}{2}(f^2-g^2)\right]\left[\frac{i}{2\tau f_1^3(ik)}\right]^2\left[\frac{f_2^8(ik)}{f_1^2(ik)}\right]\times\\ &\quad\times\left[1+\left(\frac{1}{6}+\tau^2 k\frac{\partial}{\partial k}\log f_1^2(ik)\right)(f^2-g^2)\right]\left[1+2(f^2-g^2)\frac{\tau^2}{\pi}\partial_\tau\log\left(f_1(ik)f_2^2(ik)\right)\right]\\ &= \mp\frac{1}{(8\pi^2\alpha')^2}\int d^4x\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\left(\frac{f_2(ik)}{f_1(ik)}\right)^8\times\\ &\quad\times\left[\frac{1}{\tau^2}+\left(-\frac{1}{3\tau^2}+\frac{2}{\pi}\partial_\tau\log f_2^2(ik)\right)(f^2-g^2)\right].\end{aligned} \tag{A.13}$$

From it we can easily obtain eq. (3.40) if we take into account that $f(g) = 2\pi\alpha'\hat{f}(\hat{g})$.

In order to write the interaction in eq. (3.33) in the closed channel we need to perform the modular transformation $\tau = 1/4t$ that gives

$$\begin{aligned}Z^c(F)_{\Omega' I_6} &= \mp\frac{1}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty\frac{dt}{t^3}\sin\pi\nu_f\sin\pi\nu_g\times\\ &\quad\times\frac{f_2^4(ie^{-\pi t})\Theta_2\left(i\frac{\nu_f}{4t}|\frac{i}{4t}+\frac{1}{2}\right)\Theta_2\left(i\frac{\nu_g}{4t}|\frac{i}{4t}+\frac{1}{2}\right)}{f_1^4(ie^{-\pi t})\Theta_1\left(i\frac{\nu_f}{4t}|\frac{i}{4t}+\frac{1}{2}\right)\Theta_1\left(i\frac{\nu_g}{4t}|\frac{i}{4t}+\frac{1}{2}\right)}.\end{aligned} \tag{A.14}$$

In the following we will write the formulas for the Θ-functions that are needed to get the previous equation and to show that it is equal to eq. (3.41).

Under an arbitrary modular transformation $t \to \frac{at+b}{ct+d}$, Θ_1 transforms as follows

$$\Theta_1 \left(\frac{\nu}{ct+d} | \frac{at+b}{ct+d} \right) = \eta' \, \Theta_1(\nu|t) e^{i\pi c\nu^2/(ct+d)} (ct+d)^{\frac{1}{2}} \,, \tag{A.15}$$

where η' is an eight root of unity. It implies the following transformation of Θ_1:

$$\Theta_1 \left(-\frac{\nu}{2} | \frac{t}{4} - \frac{1}{2} \right) = \frac{1}{\eta'} \Theta_1 \left(-\frac{\nu}{t} | \frac{1}{2} - \frac{1}{t} \right) e^{-i\pi\nu^2/t} \left(\frac{2}{t} \right)^{\frac{1}{2}} \tag{A.16}$$

that is obtained from eq. (A.16) by first making the substitutions $\nu \to -\frac{\nu}{2}$ and $t \to \frac{t}{4} - \frac{1}{2}$ and then choosing $a = d = 1, c = 2$ and $b = 0$. By performing in the previous equation the substitution $t \to 4it$ we get the following equation:

$$\Theta_1 \left(-\frac{\nu}{2} | it - \frac{1}{2} \right) = \frac{1}{\eta'} \Theta_1 \left(\frac{i\nu}{4t} | \frac{1}{2} + \frac{i}{4t} \right) e^{-\pi\nu^2/(4t)} \left(\frac{1}{2it} \right)^{\frac{1}{2}} \,. \tag{A.17}$$

Finally, by using eq. (A.15) with $a = b = d = 1$ and $c = 0$ we can write:

$$\Theta_1 \left(\nu | t+1 \right) = \eta' \Theta_1 \left(\nu | t \right) . \tag{A.18}$$

The latter allows us to write

$$\Theta_1 \left(-\frac{\nu}{2} | it - \frac{1}{2} \right) = \frac{1}{\eta'} \Theta_1 \left(-\frac{\nu}{2} | it + \frac{1}{2} \right) \tag{A.19}$$

that inserted in eq. (A.17) leads to

$$\Theta_1 \left(-\frac{\nu}{2} | it + \frac{1}{2} \right) = \Theta_1 \left(\frac{i\nu}{4t} | \frac{1}{2} + \frac{i}{4t} \right) e^{-\pi\nu^2/(4t)} \left(\frac{1}{2it} \right)^{\frac{1}{2}} \,. \tag{A.20}$$

In order to get the analogous transformation property of Θ_2 we use the following relation:

$$\Theta_2 \left(-\frac{\nu}{2} | \frac{t}{4} - \frac{1}{2} \right) = -\Theta_1 \left(\frac{1-\nu}{2} | \frac{t}{4} - \frac{1}{2} \right) . \tag{A.21}$$

Then, by applying eq. (A.15) with $\nu \to \frac{1-\nu}{2}$, $t \to \frac{t}{4} - \frac{1}{2}$ and $a = d = 1$, $c = 2$, $b = 0$, to the second term of the previous equation, it can be rewritten as

$$\Theta_1 \left(\frac{1-\nu}{2} | \frac{t}{4} - \frac{1}{2} \right) = \frac{1}{\eta'} \left(\frac{2}{t} \right)^{1/2} e^{-\frac{i\pi(1-\nu)^2}{t}} \Theta_1 \left(\frac{1-\nu}{t} | \frac{1}{2} - \frac{1}{t} \right) \tag{A.22}$$

and then substituting it in eq. (A.21) and performing the substitution $t \to 1/t$, one gets:

$$\Theta_2 \left(-\frac{\nu}{2} | \frac{1}{4t} - \frac{1}{2} \right) = -\frac{1}{\eta'} (2t)^{1/2} e^{-i\pi(1-\nu)^2 t} \Theta_1 \left((1-\nu)t \, | \frac{1}{2} - t \right) . \tag{A.23}$$

Let us consider the Θ_1 appearing in the second term of the previous equation. By defining in it $t' \equiv \frac{1}{2} - t$ and then $\nu' \equiv -\nu \left(\frac{1}{2} - t' \right)$ we can rewrite it as

$$\Theta_1 \left((1-\nu)t \, | \frac{1}{2} - t \right) = \Theta_1 \left(\nu' - t' + \frac{1}{2} \, | t' \right) . \tag{A.24}$$

Therefore, by using eq. (A.6) with $a = b = 1$, $\epsilon_2 = 1, \epsilon_1 = -2$, we can write eq. (A.24) as follows:

$$\Theta_1\left((1-\nu)t\,|\frac{1}{2}-t\right) = ie^{i\pi(1-2\nu)t}\Theta_2\left(-\nu t\,|\frac{1}{2}-t\right) \tag{A.25}$$

where we have restored the variables ν and t and used the following identity:

$$\Theta\begin{bmatrix}-1\\2\end{bmatrix}(\nu|t) = -\Theta\begin{bmatrix}1\\0\end{bmatrix}(\nu|t) = -\Theta_2(\nu|t) \tag{A.26}$$

that can be easily derived starting from the general expression of the theta-function given in eq. (A.3). Then by inserting eq. (A.25) in eq. (A.23) we get

$$\Theta_2\left(-\frac{\nu}{2}|\frac{1}{4t}-\frac{1}{2}\right) = -\frac{i}{\eta'}\Theta_2\left(-\nu t|\frac{1}{2}-t\right)e^{-i\pi\nu^2 t}(2t)^{\frac{1}{2}}. \tag{A.27}$$

Furthermore, by performing the substitution $t \to -\frac{i}{4t}$, eq. (A.27) becomes

$$\Theta_2\left(-\frac{\nu}{2}|it-\frac{1}{2}\right) = -\frac{i}{\eta'}\Theta_2\left(\frac{i\nu}{4t}|\frac{1}{2}+\frac{i}{4t}\right)e^{-\pi\nu^2/(4t)}(2it)^{-\frac{1}{2}}. \tag{A.28}$$

Finally, by rewriting Θ_2-function in terms of Θ_1 by means of eq. (A.5) and using eq. (A.18), we can write:

$$\Theta_2\left(-\frac{\nu}{2}|it-\frac{1}{2}\right) = \frac{1}{\eta'}\Theta_2\left(-\frac{\nu}{2}|it+\frac{1}{2}\right). \tag{A.29}$$

The last identity allows us to write:

$$\Theta_2\left(-\frac{\nu}{2}|it+\frac{1}{2}\right) = -i\Theta_2\left(\frac{i\nu}{4t}|\frac{1}{2}+\frac{i}{4t}\right)e^{-\pi\nu^2/(4t)}(2it)^{-\frac{1}{2}}. \tag{A.30}$$

The modular transformations of the f-functions with complex argument are:

$$f_1(ie^{-\pi s}) = (2s)^{-1/2}f_1(ie^{-\frac{\pi}{4s}}) \qquad f_2(ie^{-\pi s}) = f_2(ie^{-\frac{\pi}{4s}}) \tag{A.31}$$

$$f_3(ie^{-\pi s}) = e^{i\pi/8}f_4(ie^{-\frac{\pi}{4s}}) \qquad f_4(ie^{-\pi s}) = e^{-i\pi/8}f_3(ie^{-\frac{\pi}{4s}}). \tag{A.32}$$

Eq. (A.14) is obtained from eq. (3.33) by using eqs. (A.31) and by changing variable from τ to $t = \frac{1}{4\tau}$. Finally by using eqs. (A.20) and (A.30) one gets eq. (3.41) from eq. (A.14).

In order to obtain eq. (3.42) we have used the following expansions in the external field

$$\begin{aligned}\Theta_n\left(\frac{\nu_f}{2}|it+\frac{1}{2}\right) &\simeq \Theta_n\left[0|it+\frac{1}{2}\right] - \frac{2}{\pi}\partial_t\Theta_n\left(0|it+\frac{1}{2}\right)\frac{f^2}{4}\\ &= f_1(iq)f_n^2(iq)\left[1+2q\partial_q\log[f_1(iq)f_n^2(iq)]\,\frac{f^2}{4}\right]\end{aligned} \tag{A.33}$$

for $n = 2, 3, 4$, and

$$\frac{\sin\pi\nu_f}{\Theta_1\left(\frac{\nu_f}{2}|it+\frac{1}{2}\right)} \simeq -\frac{1}{f_1^3(iq)}\left\{1+f^2\left[\frac{1}{8}-q\partial_q\frac{1}{2}\log\prod_n\left(1-(iq)^{2n}\right)\right]\right\} \tag{A.34}$$

for Θ_1.

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References

[1] M. Bertolini et al., *Fractional D-branes and their gauge duals*, *J. High Energy Phys.* **02** (2001) 014 [hep-th/0011077].

[2] I.R. Klebanov, P. Ouyang and E. Witten, *A gravity dual of the chiral anomaly*, *Phys. Rev.* **D 65** (2002) 105007 [hep-th/0202056].

[3] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda and R. Marotta, *More anomalies from fractional branes*, *Phys. Lett.* **B 540** (2002) 104 [hep-th/0202195].

[4] M. Bertolini, P. Di Vecchia and R. Marotta, $N = 2$ *four-dimensional gauge theories from fractional branes*, hep-th/0112195.

[5] C.P. Herzog, I.R. Klebanov and P. Ouyang, *D-branes on the conifold and* $N = 1$ *gauge/gravity dualities*, hep-th/0205100.

[6] M. Bertolini, *Four lectures on the gauge-gravity correspondence*, *Int. J. Mod. Phys.* **A 18** (2003) 5647 [hep-th/0303160].

[7] F. Bigazzi, A.L. Cotrone, M. Petrini and A. Zaffaroni, *Supergravity duals of supersymmetric four dimensional gauge theories*, *Nuovo Cim.* **25N12** (2002) 1 [hep-th/0303191].

[8] P. Di Vecchia and A. Liccardo, *Gauge theories from D-branes*, hep-th/0307104.

[9] N. Ohta, *Cancellation of dilaton tadpoles and two loop finiteness in* SO(32) *type-I superstring*, *Phys. Rev. Lett.* **59** (1987) 176.

[10] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[11] P. Di Vecchia, A. Liccardo, R. Marotta and F. Pezzella, *Gauge/gravity correspondence from open/closed string duality*, *J. High Energy Phys.* **06** (2003) 007 [hep-th/0305061].

[12] M. Bianchi and J.F. Morales, *Anomalies & tadpoles*, *J. High Energy Phys.* **03** (2000) 030 [hep-th/0002149].

[13] I.R. Klebanov and A.A. Tseytlin, *D-branes and dual gauge theories in type 0 strings*, *Nucl. Phys.* **B 546** (1999) 155 [hep-th/9811035].

[14] I.R. Klebanov and A.A. Tseytlin, *Asymptotic freedom and infrared behavior in the type 0 string approach to gauge theory*, *Nucl. Phys.* **B 547** (1999) 143 [hep-th/9812089].

[15] I.R. Klebanov and A.A. Tseytlin, *A non-supersymmetric large-N CFT from type 0 string theory*, *J. High Energy Phys.* **03** (1999) 015 [hep-th/9901101].

[16] A. Sagnotti, *Some properties of open string theories*, hep-th/9509080.

[17] R. Blumenhagen, A. Font and D. Lüst, *Tachyon-free orientifolds of type 0B strings in various dimensions*, *Nucl. Phys.* **B 558** (1999) 159 [hep-th/9904069].

[18] C. Angelantonj and A. Armoni, *Non-tachyonic type 0B orientifolds, non-supersymmetric gauge theories and cosmological rg flow*, *Nucl. Phys.* **B 578** (2000) 239 [hep-th/9912257].

[19] E. Dudas and J. Mourad, *D-branes in non-tachyonic 0B orientifolds*, *Nucl. Phys.* **B 598** (2001) 189 [hep-th/0010179].

[20] A. Armoni, M. Shifman and G. Veneziano, *From super-Yang-Mills theory to QCD: planar equivalence and its implications*, hep-th/0403071.

JHEP09(2004)050

[21] F. Sannino and M. Shifman, *Effective lagrangians for orientifold theories*, *Phys. Rev.* **D 69** (2004) 125004 [hep-th/0309252].

[22] S. Kachru and E. Silverstein, *4D conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855 [hep-th/9802183];
Z. Kakushadze, *Gauge theories from orientifolds and large-N limit*, *Nucl. Phys.* **B 529** (1998) 157 [hep-th/9803214];
M. Bershadsky, Z. Kakushadze and C. Vafa, *String expansion as large-N expansion of gauge theories*, *Nucl. Phys.* **B 523** (1998) 59 [hep-th/9803076].

[23] A. Armoni, M. Shifman and G. Veneziano, *Exact results in non-supersymmetric large-N orientifold field theories*, *Nucl. Phys.* **B 667** (2003) 170 [hep-th/0302163].

[24] C. Bachas and C. Fabre, *Threshold effects in open-string theory*, *Nucl. Phys.* **B 476** (1996) 418 [hep-th/9605028].

[25] N. Nekrasov and S.L. Shatashvili, *On non-supersymmetric CFT in four dimensions*, *Phys. Rept.* **320** (1999) 127 [hep-th/9902110].

[26] R. Blumenhagen, A. Font and D. Lüst, *Non-supersymmetric gauge theories from D-branes in type 0 string theory*, *Nucl. Phys.* **B 560** (1999) 66 [hep-th/9906101].

[27] M. Frau, L. Gallot, A. Lerda and P. Strigazzi, *Stable non-BPS D-branes in type-I string theory*, *Nucl. Phys.* **B 564** (2000) 60 [hep-th/9903123].

[28] A. Liccardo, *Aspects of perturbative and non-perturbative string theory*, Ph.D thesis, unpublished.

[29] A. Lerda and R. Russo, *Stable non-BPS states in string theory: a pedagogical review*, *Int. J. Mod. Phys.* **A 15** (2000) 771 [hep-th/9905006].

[30] A. Dabholkar, *Lectures on orientifolds and duality*, hep-th/9804208.

[31] E.G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and d-manifolds*, *Phys. Rev.* **D 54** (1996) 1667 [hep-th/9601038].

[32] R. Marotta, F. Nicodemi, R. Pettorino, F. Pezzella and F. Sannino, $N = 1$ *matter from fractional branes*, *J. High Energy Phys.* **09** (2002) 010 [hep-th/0208153].

[33] M. Bertolini, P. Di Vecchia, G. Ferretti and R. Marotta, *Fractional branes and* $N = 1$ *gauge theories*, *Nucl. Phys.* **B 630** (2002) 222 [hep-th/0112187].